Optimal control and optimization for a class of repetitive processes

M. Dymkov, E. Rogers, K. Galkowski and D. H. Owens

Abstract— **This paper gives new results on optimization and optimal control of discrete linear repetitive processes. These are for the class of so-called wave processes which have recently been introduced to describe physical examples which contain dynamics not captured by previously considered models.**

I. INTRODUCTION

Repetitive processes are a distinct class of two-dimensional (2D) systems of both systems theoretic and applications interest. Their unique characteristic is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

Physical examples of repetitive processes include long-wall coal cutting and metal rolling operations (see, for example, [6]). Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications include classes of iterative learning control (ILC) schemes [1] and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle [5]. More recently, there has been work on using a repetitive process setting to study spatially distributed dynamic systems (see, for example, [2]) where causality only has physical meaning for the time variable. Note that in many applications such causality is too strong a requirement and is often not physically motivated. This is of particular interest in signal processing applications, especially those where space indeterminates occur and right upper quadrant causality is irrelevant. This has led to a so-called wave repetitive processes and this paper develops the first results on optimal control of them.

II. BACKGROUND

The most basic discrete linear repetitive process state-space model [6] has the following form over $0 \le m \le T-1$, $t \ge 0$, where $T < \infty$ is the pass length

$$\begin{aligned} x_{t+1}(m+1) &= Ax_{t+1}(m) + Bu_{t+1}(m) + B_0 y_t(m) \\ y_{t+1}(m) &= Cx_{t+1}(m) + Du_{t+1}(m) + D_0 y_t(m) \end{aligned} \tag{1}$$

Here on pass t, $x_t(m) \in \mathbb{R}^n$ is the state vector, $y_t(m) \in \mathbb{R}^\mu$ is the pass profile vector, and $u_t(m) \in \mathbb{R}^r$ is the vector of control inputs. The boundary conditions (i.e. the pass state initial vector sequence and the initial pass profile) are

$$\begin{aligned} x_{t+1}(0) &= d_{t+1}, & t \ge 0 \\ y_0(m) &= f(m), & 0 \le p \le T-1 \end{aligned} \tag{2}$$

where the $n \times 1$ vector d_{t+1} has known constant entries and $f(m)$ is an $m \times 1$ vector whose entries are known functions of p.

This model is for the case where at any point m on pass t the only previous pass contribution arises from $y_{t-1}(m)$ — the so-called point property. In some practical applications, however, this may not be a realistic assumption. For example, in long-wall coal cutting the pass profile is the height of the stone/coal interface above some datum line. Also the cutting machine is mounted on the so-called armored face conveyor and during the production of the current pass rests on that cut during the previous pass. At the end of each pass, the machine is hauled back (at relatively high speed) and then hydraulic rams are used to push the complete installation over onto the newly cut pass profile before the start of the next pass. Given that the weight here can be up to 5 tonnes, it is clear that the previous pass profile will, in general, be substantially modified before the next pass begins (and also during it). This is termed inter-pass smoothing and clearly there will be cases when it is essential to include a representation of its effects in any analysis.

In general, the question of how to represent inter-pass smoothing is a non-trivial question. Here we consider the case of discrete along the pass dynamics and assume that all points along the previous pass will contribute to the current pass profile vector at any point. The collection of points from the previous pass which explicitly contribute to the current pass profile vector at any point along the pass is termed a window. These features are included in the following model [3] over $t \ge 0$ and $0 \le m \le T - \overline{\gamma}$ (also termed a wave (or extended wave) discrete linear repetitive process)

$$x_{t+1}(m) = \sum_{i=-\underline{\gamma}}^{\overline{\gamma}} \left(A_i x_t(m+i) + B_i u_t^i(m)\right) \tag{3}$$

M. Dymkov is with the Department of Mathematics, Belarus State Economic University, Partizanskiy Ave 26, 220070 Minsk, Belarus dymkov_m@bseu.by

E. Rogers is with the School of Electronics and Computer Science University of Southampton Southampton, SO17 1BJ, UK etar@ecs.soton.ac.uk

K. Galkowski is with the Institute of Control and Computation Engineering, University of Zielona Gora, Poland galkowsk@uni-wuppertal.de

D.H. Owens is with the Department of Automatic Control and Systems Engineering, University of Sheffield, Sheffield S1 3JD. UK

1-4244-1112-2/07/$25.00 ©2007 IEEE

where $\underline{\gamma}, \overline{\gamma} \in \mathbb{Z}_+$. The associated boundary conditions are

$$\begin{aligned} x_t(\gamma) &= 0, \qquad -\underline{\gamma} \le \gamma < 0,\ t \ge 0 \\ x_0(m) &= d(m), \quad 0 \le m < T \\ x_t(T-\gamma) &= d_t(\gamma), \qquad 0 \le \gamma < \overline{\gamma},\ t \ge 0 \end{aligned} \tag{4}$$

where $d(m)$ is an $n \times 1$ real valued vector function of m over $0 \le m \le T$ and, for each γ such that $0 \le \gamma < \overline{\gamma}$, the vector sequence $\{d_t(\gamma)\}_{t \ge 0}, d_t \in \mathbb{R}^n$ is assumed to be bounded.

More generally, all control inputs along the previous pass may also contribute to the pass profile vector at any point on the current pass. Also, the so called central property may be present, i.e. the dynamics evolve over $[-T, T]$ instead of $[0, T]$. The model considered here is

$$\begin{aligned} x(t+1,m) &= \sum_{i=-N}^{N} A_i(t, m+i) + Bu(t,m) \\ t &= 0,1...,T,\ m \in -N,\dots,0,\dots N \end{aligned} \tag{5}$$

with the given initial conditions

$$x(0,m) = \phi(m),\ m \in -N,\dots,0,\dots N \tag{6}$$

where $x(t,m) \in \mathbb{R}^n,\ u(t,m) \in \mathbb{R}^r$ for any (t,m). It is also possible to consider the cases when $T \to \infty$ and $N \to \infty$.

III. Analysis

Consider first the following optimization problem: find the admissible control $u^0(t,m)$ which minimizes the cost function

$$J(u) = \sum_{t=0}^{T} \sum_{m=-\infty}^{\infty} [Qx^2(t,m) + Ru^2(t,m)] \tag{7}$$

over the solutions of (5) with $N = \infty$, i.e. the infinite pass number case. Also, it is assumed that the matrices A_i satisfy

$$\sum_{i=-\infty}^{+\infty} (1+\varepsilon)^i \|A_i\| < \infty \tag{8}$$

for some real number $\varepsilon > 0$, which guarantees the convergence of the series $\sum_{i=-\infty}^{+\infty} z^i A_i$ in a domain including the unit disc of complex plane $\mathbb{C}$.

Let $l^2(\mathbb{R}^n)$ and $l^2(\mathbb{R}^m)$ denote the spaces of the square summable sequences from $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Also introduce

$$u^0 = \{u_t^0,\ t = 0,1,\dots,T\},\ u^0 \in (l^2(\mathbb{R}^m))^{T+1}$$

where

$$u_t^0 = \{u^0(t,0), u(t,1), \dots\} \in l^2(\mathbb{R}^m) \quad \forall t$$

and $\phi = \{\phi(0), \phi(1), \dots\} \in l^2(\mathbb{R}^n)$. Then it is straightforward to show [4] that the optimization problem considered here has a unique solution which can be written in operator form as

$$u^0 = -(\mathscr{R} + L^*\mathscr{Q}L)^{-1} L^* \mathscr{Q}\phi \tag{9}$$

where $L: (l^2(\mathbb{R}^m))^{T+1} \to (l^2(\mathbb{R}^n))^{T+1}$ is the operator defined by

$$\begin{aligned} (L\gamma)_t &= \mathscr{B}\gamma_{t-1} + \mathscr{A}\mathscr{B}\gamma_{t-2} + \dots + \mathscr{A}^{t-1}\mathscr{B}\gamma_0 \\ (L\gamma)_0 &= 0,\ t > 0 \end{aligned} \tag{10}$$

the operator $\mathscr{A} : l^2(\mathbb{R}^n) \to l^2(\mathbb{R}^n)$ is given by

$$(\mathscr{A}\alpha)(m) = \sum_{i=-\infty}^{+\infty} A_i \alpha(m+i),\ m \in \mathbb{Z} \tag{11}$$

where the operator $\mathscr{B}$ is directly generated by B and the operators $\mathscr{R}$ and $\mathscr{Q}$ as

$$\begin{aligned} (\mathscr{R}u)(t,m) &= Ru(t,m),\ t \neq 0 \\ (\mathscr{Q}x)(t,m) &= Qx(t,m), t = 0,\dots,T,\ m \in \mathbb{Z} \end{aligned}$$

The adjoint operator L^* is defined (as usual) by

$$\begin{aligned} (L^*\beta)_t &= \mathscr{B}^*\beta_{t+1} + \mathscr{B}^*\mathscr{A}^*\gamma_{t+2} \\ &+ \dots + \mathscr{B}^*\mathscr{A}^{*T-t-1}\gamma_T \end{aligned} \tag{12}$$

and the adjoint operator $\mathscr{A}^* : l^2(\mathbb{R}^n) \to l^2(\mathbb{R}^n)$ is

$$(\mathscr{A}^*)(\psi) = \sum_{i=-\infty}^{+\infty} A_i^* \psi(m-i)$$

where A_i^* denotes the conjugate transpose. Note also that, since $\mathscr{G} \ge 0$, $\mathscr{R} > 0$, the operator $\mathscr{R} + L^*\mathscr{Q}L$ is invertible. Clearly, however, (9) is not suitable for implementation.

We will use the following result in the analysis below. Its proof is routine and hence is omitted here.

Theorem 1: The system

$$x(t+1,m) = \sum_{i=-\infty}^{\infty} A_i(t, m+i) - BR^{-1}B^*z(t,m) \tag{13}$$

$$z(t,m) = \sum_{i=-\infty}^{\infty} A_i^* z(t+1, m-i) + Qx(t+1,m), \tag{14}$$

$$(t,m) \in \{0,\dots,T\} \times \mathbb{Z}$$

with

$$x(0,m) = \varphi(m),\ z(T,m) = 0,\ m \in \mathbb{Z} \tag{15}$$

has a solution in l_2.

Theorem 2: The optimal control problem defined by (5)–(7) has unique solution

$$u^0(t,m) = -R^{-1}B^*z(t,m),\ t \in \{0,\dots,T\},\ m \in \mathbb{Z}$$

where $z(t,s)$ is the solution of (13)—(15).

Proof. Note first that the uniqueness of the optimal control has already been established — see (9). Now let $(x(t,m), z(t,m))$, $t \in \{0,\dots,T\}$, $m \in \mathbb{Z}_+$, be the solution of (13)–(15) and introduce

$$\hat{u}(t,m) = -R^{-1}B^*z(t,m),\ t \in \{0,\dots,T\},\ m \in \mathbb{Z}_+$$

Then we can rewrite (13)–(15) as

$$\begin{aligned} y_{t+1} &= \mathscr{A}y_t - \mathscr{B}\mathscr{R}^{-1}\mathscr{B}^*w_t \\ w_t &= \mathscr{A}^*w_{t+1} + \mathscr{Q}y_{t+1}, y_0 = \varphi,\ w_T = 0 \end{aligned} \tag{16}$$

and it follows immediately that

$$
\begin{aligned}
y_t &= \mathscr{A}^t\varphi - \sum_{i=0}^{t-1}\mathscr{A}^i\mathscr{B}\mathscr{R}^{-1}\mathscr{B}^* w_{t-1-i},\ t\in\{0,\dots,T\}\\
w_t &= \sum_{i=0}^{T-t}\mathscr{A}^{*i}\mathscr{Q}y_{t+i},\quad \hat{v}_t = \mathscr{R}^{-1}\mathscr{B}^* w_t \qquad (17)
\end{aligned}
$$

where $(\hat{v}_t)(s) = \hat{u}(t,s)$. Using the result of the previous theorem, it now follows that $\hat{v} = (\hat{v}_0,\dots,\hat{v}_T)$ can be written in the form

$$\hat{v} = -\mathscr{R}^{-1}L^*\mathscr{Q}y$$

and hence

$$\mathscr{R}\hat{v} = -L^*\mathscr{Q}y$$

Conversely, we have that

$$y = \omega - L\mathscr{R}^{-1}\mathscr{B}^* w \text{ or } y = \omega + L\hat{v}$$

Therefore

$$
\begin{aligned}
\mathscr{R}\hat{v} &= -L^*\mathscr{Q}y + L^*\mathscr{Q}\omega - L^*\mathscr{Q}\omega\\
&= L^*\mathscr{Q}(\omega - y) - L^*\mathscr{Q}\omega = -L^*\mathscr{Q}L\hat{v} - L^*\mathscr{Q}\omega
\end{aligned}
$$

and

$$\hat{v} = -(\mathscr{R} + L^*\mathscr{Q}L)^{-1}L^*\mathscr{Q}\omega$$

Hence $\hat{v}$ coincides with u^0 defined by (9). Consequently $\hat{u}(t,s) = u^0(t,s) = -\mathscr{R}^{-1}\mathscr{B}^* z(t,s)$ is the solution of the optimal control problem and the proof is complete. ■

A. Optimal feedback control

In this section we further develop the analysis of the previous section to produce an optimal control in feedback form. In particular, suppose that $P_t : l^2(\mathbb{R}^n) \to l^2(\mathbb{R}^n), t = 1,\dots,T-1,\ P_T = 0$ is a family of linear operators and let u^0 be the optimal control for (5)–(7) and x^0 the corresponding trajectory. Then the problem here is to construct $P_t, t\geq 0$, such that

$$u_t^0 = -\mathscr{R}^{-1}\mathscr{B}^* P_t x_t^0, t = 1,\dots,T-1 \qquad (18)$$

Theorem 3: If the optimal feedback control problem considered here has a solution then the operators P_t satisfy the following equations

$$P_{t-1} + (\mathscr{L} \quad + \quad \mathscr{A}^* P_t)\mathscr{B}\mathscr{R}^{-1}\mathscr{B}^* P_{t-1} \qquad (19)$$

$$= \quad (\mathscr{L} + \mathscr{A}^* P_t)\mathscr{A},\ P_T = 0, t\geq 0 \quad (20)$$

Moreover, the minimal cost value is $J^0 = (P_0\varphi, \mathscr{A}\varphi)$ where $(\cdot,\cdot)$ denotes the corresponding scalar product.

Proof. Suppose operators exist such that (18) holds. Then the function x^0 satisfies

$$
\begin{aligned}
y_{t+1} &= (\mathscr{A} - \mathscr{B}\mathscr{R}^{-1}\mathscr{B}^*\mathscr{P}_t)y_t,\ t\in\{0,\dots,T\}\\
y_0 &= \varphi \qquad (21)
\end{aligned}
$$

since if we substitute $u^0 = -\mathscr{R}^{-1}\mathscr{B}^*\mathscr{P}_t y_t^0$ in (5) then in operator form we obtain

$$y_{t+1} = \mathscr{A}y_t + \mathscr{B}u_t = \mathscr{A}y_t - \mathscr{B}\mathscr{R}^{-1}\mathscr{B}^*\mathscr{P}_t y_t$$

$$= (\mathscr{A} - \mathscr{B}\mathscr{R}^{-1}\mathscr{B}^*\mathscr{P}_t)y_t$$

Conversely, the solution of (21) can be rewritten as

$$y_t^0 = F_{t-1}(F_{t-2}\dots(F_s y_s^0))\ \forall\ t > s\geq 0$$

where

$$F_t = \mathscr{A} - \mathscr{B}\mathscr{R}^{-1}\mathscr{B}^*\mathscr{P}_t$$

Hence we have the system $y_{t+1} = F_t y_t$ with initial conditions $y_s = y_s^0$, where $s > 0$ is an arbitrary index. Substituting this into $u^0 = -\mathscr{R}^{-1}L^*\mathscr{Q}y^0$ completes the first part of the proof.

Now we have that

$$L^*(Qy^0)_t = \mathscr{B}^*(Qy^0)_{t+1} + \dots + \mathscr{B}^*\mathscr{A}^{*(T-t-1)}(Qy^0)_T =$$
$$\mathscr{B}^*[(Qy^0)_{t+1} + \mathscr{A}^*(Qy^0)_{t+2} + \dots + \mathscr{A}^{*(T-t-1)}(Qy^0)_T]$$

and hence

$$
\begin{aligned}
-\mathscr{R}^{-1}\mathscr{B}^*\mathscr{P}_t y_t^0 &= v^0 = -\mathscr{R}^{-1}L^*\mathscr{Q}y^0 =\\
&- \mathscr{R}^{-1}\mathscr{B}^*[(Qy^0)_{t+1} + \mathscr{A}^*(Qy^0)_{t+2}\\
&+ \dots + \mathscr{A}^{*(T-t-1)}(Qy^0)_T]
\end{aligned}
$$

Also the solutions y_{t+1}^0 can be represented by the system $y_{t+1} = F_t y_t$, and start with an arbitrary index s as initial data to obtain

$$
\begin{cases}
y_{t+1}^0 &= F_t y_t^0\\
y_{t+2}^0 &= F_{t+1}(F_t y_t^0)\\
&\dots\\
y_T^0 &= F_{T-1}(F_{T-2}(\cdots F_t y_t^0))
\end{cases}
$$

Then

$$\mathscr{P}_t y_t^0 = QF_t y_t^0 + \mathscr{A}^* QF_{t+1}F_t y_t + \cdots$$
$$+\mathscr{A}^{*(T-t-1)}QF_{T-1}F_{T-2}\cdots F_t y_t$$

or

$$\mathscr{P}_t y_t^0 = (QF_t + \mathscr{A}^* QF_{t+1}F_t + \cdots$$
$$+\mathscr{A}^{*(T-t-1)}QF_{T-1}F_{T-2}\cdots F_t)y_t$$

Consequently the required operators $\mathscr{P}_t$ satisfy

$$
\begin{aligned}
\mathscr{P}_{t-1} &= \mathscr{Q}F_{t-1} + \mathscr{A}^*\mathscr{Q}F_t F_{t-1} + \dots\\
&+ \mathscr{A}^{*T-t}\mathscr{Q}F_{T-1}\dots F_{t-1} \qquad (22)
\end{aligned}
$$

with $\mathscr{P}_T = 0$, or in recurrent form

$$
\begin{aligned}
\mathscr{P}_{t-1} &+ (\mathscr{Q} + \mathscr{A}^*\mathscr{P}_t)\mathscr{B}\mathscr{R}^{-1}\mathscr{B}^*\mathscr{P}_{t-1}\\
&= (\mathscr{Q} + \mathscr{A}^*\mathscr{P}_t)\mathscr{A}, t = 1,\dots,T,\ \mathscr{P}_T = 0
\end{aligned}
\qquad (23)
$$

Now let $\mathscr{P}_t^0$ be a solution of (19). Then we have that

$$(\mathscr{P}_{t-1}^0 y_{t-1}^0, \mathscr{A}y_{t-1}^0) - (\mathscr{P}_t y_t^0, \mathscr{A}y_t^0)$$
$$= (\mathscr{P}_{t-1}^0 y_{t-1}^0, \mathscr{A}y_{t-1}^0)$$
$$-(\mathscr{A}^*\mathscr{P}_t^0(\mathscr{A} - \mathscr{B}\mathscr{R}^{-1}\mathscr{B}^*\mathscr{P}_{t-1}^0)y_{t-1}^0, y_t^0) =$$
$$(\mathscr{P}_{t-1}^0 y_{t-1}^0, \mathscr{A}y_{t-1}^0) - (\mathscr{P}_{t-1}^0 y_{t-1}^0, y_{t-1}^0)$$
$$+(\mathscr{Q}(\mathscr{A} - \mathscr{B}\mathscr{R}^{-1}\mathscr{B}^*\mathscr{P}_{t-1}^0)y_{t-1}^0, y_t^0) =$$
$$(\mathscr{P}_{t-1}^0 y_{t-1}^0, \mathscr{B}\mathscr{R}^{-1}\mathscr{B}^*\mathscr{P}_{t-1}^0)y_{t-1}^0)+$$

$$(\mathscr{Q}y_t^0, y_t^0) = (\mathscr{R}v_{t-1}^0, v_{t-1}^0) + (\mathscr{Q}y_t^0, y_t^0)$$

Hence

$$J(v^0) = \sum_{t=1}^{T} (\mathscr{Q}y_t^0, y_t^0) + (\mathscr{R}v_{t-1}^0, v_{t-1}^0) =$$

$$\sum_{t=1}^{T} \left[(\mathscr{P}_{t-1}^0 y_{t-1}^0, \mathscr{A} y_{t-1}^0) - (\mathscr{P}_t^0 y_t^0, \mathscr{A} y_{t-1}^0) \right]$$

$$= (\mathscr{P}_0 y_0^0, \mathscr{A} y_0^0) = (\mathscr{P}_0^0 \varphi, \mathscr{A} \varphi)$$

and the proof is complete. ■

B. Optimal control for $T \to \infty$

We now consider the problem (5)–(7) for the case when $T \to \infty$. In particular, let l_2^2 be the space of all sequences $\{v(t,m)\}$ where $(t,m) \in \mathbb{Z}_+ \times \mathbb{Z}$ and such that

$$\sum_{(t,m) \in \mathbb{Z}_+ \times \mathbb{Z}} | v(t,m) |^2 < \infty$$

Suppose also that the operators A_i are such that the spectral radius, denoted by $r(\cdot)$, of the operator of (11) satisfies $r(\mathscr{A}) < 1$. Then we have the following result whose proof follows by routine extension of that for Theorem 2 and hence is omitted here.

Theorem 4: Assume that $T \to \infty$. Then the optimal control problem (5)–(7) has solution

$$u_t^0 = -\mathscr{B}^* P x_t^0, \ t > 0 \tag{24}$$

where $x_t^0, t \in \mathbb{Z}_+$ is the unique solution of

$$x_{t+1} = (\mathscr{A} - \mathscr{B}\mathscr{B}^* P) x_t, \ x_0 = \varphi \tag{25}$$

and $P : l^2(E) \to l^2(E)$ is the linear bounded operator given by

$$P = (\mathscr{R} + \mathscr{A}^* P)(\mathscr{A} - \mathscr{B}\mathscr{B}^* P) \tag{26}$$

Also, the minimum cost value is $I^0 = (P\varphi, \mathscr{A}\varphi)$.

The optimal solution for the problem (5)–(7) in this last result can also be expressed in Fourier transform terms as follows, where the discrete Fourier transform of, for example, the control is given by

$$\mathscr{U}_t(\omega) = \sum_{s=-\infty}^{\infty} u^0(t,s) e^{-js\omega}, \omega \in [0, 2\pi], j^2 = -1 \tag{27}$$

Theorem 5: The optimal control $u^0(t,s)$ (with $T \to \infty$) of Theorem 4 can be written in Fourier transform terms as

$$\mathscr{U}_t(\omega) = K(\omega) X_t(\omega),$$

where $X_t(\omega)$ denotes the Fourier transformation of the optimal trajectory $x^0(t,s)$ where

$$\begin{aligned} K(\omega) &= -[R + B^* P(\omega) B]^{-1} B^* P(\omega) A(\omega) \\ A(\omega) &= \sum_{k=-\infty}^{+\infty} e^{jk\omega} A_k \end{aligned}$$

and $P(\omega)$, $\omega \in [0, 2\pi]$ is given by

$$P(\omega) = Q + A^*(\omega) P(\omega) A(\omega) -$$

$$A^*(\omega) P(\omega) B [R + B^* P(\omega) B]^{-1} B^* P(\omega) A(\omega) \tag{28}$$

Also, the minimal cost value is

$$J(u^0) = \frac{1}{2\pi} \int_0^{2\pi} (X_0(\omega), P(\omega) X_0(\omega)) d\omega$$

Proof. Taking the discrete Fourier transformation of (5) with respect to the variable m, i.e.

$$X_t(\omega) = \sum_{m \in \mathbb{Z}} x^0(t,m) e^{-jm\omega}, \ \omega \in [0, 2\pi]$$

gives

$$X_{t+1}(\omega) = A(\omega) X_t(\omega) + B U_t(\omega),$$

$$A(\omega) = \sum_{k=-\infty}^{\infty} e^{ik\omega} A_k, \omega \in [0, 2\pi]$$

and hence, by Parseval's identity,

$$\begin{aligned} J(u) &= \frac{1}{2\pi} \sum_{t \in Z_+} \int_0^{2\pi} (X_t(\omega), Q X_t(\omega) X_T(\omega)) \\ &+ (U_t(\omega), R U_t(\omega)) d\omega \end{aligned}$$

Now let $P(\omega)$, $\omega \in [0, 2\pi]$ be an arbitrary collection of non-negative operators from $\mathbb{C}^n$ to $\mathbb{C}^n$ such that $\int_0^{2\pi} \|P(\omega)\| d\omega < \infty$. Then

$$\begin{aligned} 0 &= (P(\omega) X_0(\omega), X_0(\omega) - \sum_{t \in Z_+} (P(\omega) X_t(\omega), X_t(\omega)) \\ &+ \sum_{t \in Z_+} (P(\omega)(A(\omega) P(\omega) X_t(\omega) \\ &+ B U_t(\omega)), (A(\omega) P(\omega) X_t(\omega) + B U_t(\omega)) \end{aligned} \tag{29}$$

Also integrating this last identity over $\omega \in [0, 2\pi]$ and adding the result to J yields

$$\begin{aligned} J(u^0) &= \frac{1}{2\pi} \int_0^{2\pi} [(P(\omega) X_0(\omega), X_0(\omega)) \\ &+ \sum_{t \in Z_+} (Q X_t(\omega), X_t(\omega)) \\ &- (P(\omega) X_t(\omega), X_t(\omega)) + (R U_t(\omega), U_t(\omega)) \\ &+ (P(\omega) A(\omega) X_t(\omega), A(\omega) X_t(\omega)) \\ &+ (P(\omega) B U_t(\omega), B U_t(\omega)) \\ &+ (P(\omega) A(\omega) X_t(\omega), W] d\omega \end{aligned} \tag{30}$$

$$W = A(\omega) X_t(\omega), A(\omega) X_t(\omega))$$

or, on adding and subtracting, the term

$$(P(\omega) A(\omega) X_t(\omega), B[R + B' P(\omega) B]^{-1} B' P(\omega) A(\omega) X_t(\omega))$$

$$J(u) = \frac{1}{2\pi} \int_0^{2\pi} [(P(\omega) X_0(\omega), X_0(\omega)) +$$

$$\sum_{t \in Z_+} [(F(\omega) X_t(\omega), X_t(\omega)) +$$

$$((R + B' P(\omega) B) V_t(\omega), V_t(\omega))] d\omega.$$

where

$$F(\omega) = Q - P(\omega) + A^*(\omega)P(\omega)A(\omega) -$$
$$-A^*(\omega)P(\omega)B[R + B^{'}P(\omega)B]^{-1}B^{'}P(\omega)A(\omega)$$
$$V_t(\omega) = U_t(\omega) + [R + B^{'}P(\omega)B]^{-1}B^{'}P(\omega)A(\omega)X_t(\omega)$$

where the inverses required exist since the operators $P(\omega)$ and R are nonnegative and positive respectively. The second term in J does not depend on control function since $X_0(\omega) = \sum\limits_{s\in\mathbb{Z}} \varphi(s)e^{-is\omega},\ \omega \in [0, 2\pi]$.

Now chose $P(\omega)$ such that $F(\omega) = 0$. Then the cost function can be rewritten as

$$J(u) = \frac{1}{2\pi}\int_0^{2\pi}[(P(\omega)X_0(\omega), X_0(\omega)) +$$
$$\sum_{t=0}^{+\infty}([R + B'P(\omega)B]^{-1}V_t(\omega), V_t(\omega))d\omega \qquad (31)$$

and it is easy to see that its minimum value is

$$J(u^0) = \frac{1}{2\pi}\int_0^{2\pi}[(P(\omega)X_0(\omega), X_0(\omega))d\omega$$

which is feasible if, and only if, $V_t(\omega) = 0$, i.e. if, and only if, $U_t(\omega) = K(\omega)X_t(\omega)$. The expression for the optimal $K(\omega)$ and $P(\omega)$ now follow immediately and the proof is complete. ■

We now have the following result concerning the feedback control solution to the problem considered here.

Theorem 6: The optimal feedback control for (6)–(7) in the case $T \to \infty$ is given by

$$u(t,s) = \sum_{j\in-\infty}^{+\infty} K_j x(t, s+j) \qquad (32)$$

where $\{K_i,\ i \in \mathbb{Z}\}$ is a set of $n \times n$-matrices.

Proof. It is routine to show that (28) admits the solution $P = P(z)$ which is analytic in a domain including the unit disc of complex plane $\mathbb{C}$, and hence the function $K = K(z)$ also has this property in the same domain. Consequently $\exists\, \varepsilon > 0$ such that $K(z)$ can be expanded as

$$K(z) = \sum_{j=-\infty}^{+\infty} K_j z^j, \quad 1 - \varepsilon < |z| < 1 + \varepsilon$$

and hence

$$U_t(\omega) = K(\omega)X_t(\omega) = \sum_{j=-\infty}^{+\infty} K_j e^{ij\omega}X_t(\omega)$$

Taking the inverse Fourier transform of $X_t(\omega)$ now gives

$$u^0(t,s) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{j=-\infty}^{+\infty} K_j e^{ijw}X_t(w)e^{isw}dw$$

$$= \sum_{j=-\infty}^{+\infty} K_j \frac{1}{2\pi}\int_{-\pi}^{\pi} X_t(w)e^{i(s+j)w}dw = \sum_{j=-\infty}^{+\infty} K_j x(t, s+j)$$

where the matrices K_j are the coefficients of the series expansion of $K(z)$ and the proof.

IV. CONCLUSIONS

This paper has developed the first results on an optimal control/optimization theory for wave discrete linear repetitive process which have been recently introduced to model new classes of systems with repetitive dynamics which are not captured in previously used models. These systems have many practical applications and represent major new challenges for repetitive process control theory and associated control law design. There are many open questions including the question of how to turn the feedback control based results here into computational algorithms. This and other areas are the subject of ongoing research.

REFERENCES

[1] N. Amann, D. H. Owens, and E. Rogers. Predictive optimal iterative learning control, *International Journal of Control*, vol. 69(2), pp. 203–226, 1998.

[2] P. Augusta, and Z. Hurak, Multidimensional transfer function model of a deformable mirror in adaptive optics systems, *Proceedings of the 17th International Symposium on the Mathematical Theory of Networks and Systems*, pp. 1556–1561, 2006.

[3] K. Galkowski, B. Cichy, E. Rogers E., J. Lam. Stabilization of a class of uncertain "wave" discrete linear repetitive processes, in *Proceedings of the 2006 IEEE International Conference of Decission and Control*, San Diego, USA, 2006.

[4] M. P. Dymkov. Optimization Theory for Multidimensional Control Systems, *Belarus State Economic Uneversty Press*, Minsk, Belarus, 2005, (in Russian).

[5] P. D. Roberts. Numerical investigations of a stability theorem arising from 2-dimensional analysis of an iterative optimal control algorithm, *Multidimensional Systems and Signal Processing*, vol. 11(1/2), pp. 109–124, 2000.

[6] E. Rogers, K. Galkowski and D. H. Owens. Control Systems Theory and Applications for Linear Repetitive Processes, ser. Lecture Notes in Control and Information Sciences. Springer-Verlag, vol. 349, 2007.