AN APPROAOCH TO CONTROLLABILITY AND OPTIMIZATION PROBLEMS FOR REPETITIVE PROCESSES

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Abstract: In the paper the structural links between the differential-algebraic delayed systems and some classes of repetitive processes have been investigated. In particular, the pointwise completeness and associated controllability notions with respect to initial data are considered. Also the optimal control law for the fastest possible driving the process dynamics to zero equilibrium state subject to an integral control constraint is established.

1. INTRODUCTION

Differential repetitive processes are a class of continuous-discrete 2D linear systems of both systems theoretic and applications interest. Applications areas include iterative learning control and iterative solution algorithms for classes of dynamic nonlinear optimal control problems based on the maximum principle (Rogers and Owens, 1992). It is already known that repetitive processes can be represented in various dynamical system forms, which can, where appropriate, be used to great effect in the control related analysis of these processes. In this paper, we investigate further the already known links between some classes of linear repetitive processes and delay systems (Dymkov *et al*, 2003a) and apply this to analyze control theory problems arising in optimal control of these repetitive processes. In particular, we introduce various controllability notions that play a significant role for applications area. Here it should also be noted there exist more than one distinct controllability notion for the repetitive processes and delay systems considered here, see e. g. Manitius and Triggiani, 1978; Dymkov *et al*, 2003b. For the differential systems with retarded arguments and the hybrid differential- difference systems, in particular, a key role plays the so-called pointwise completness notion (Weiss, 1976; Metelskii, 1994) that leads to the controllability with respect to initial data, in fact. Also we adopt the method (Gabasov and Kirillova, 1976) based on the separation theorem for convex sets to establish optimality conditions for the time optimal control problem, which then is detailed for the special case of the control constraint to obtain more flexible form of the optimal control function. It has been conjectured that such a setting is appropriate for development the numerical methods for optimal control problems and related studies on for which very little work has been reported to date. Some areas for further research are also briefly discussed.

2. A DELAY SYSTEMS MODEL FOR DIFFERENTIAL LINEAR REPETITIVE PROCESSES

A differential linear repetitive processes (Rogers and Owens, 1992) is defined over $0 \le t \le \hat{\alpha}$, $k \ge 0$, by a state space model of the form

$$\begin{aligned} \dot{x}_{k+1}(t) =& \hat{A}x_{k+1}(t) + \hat{B}u_{k+1}(t) + \hat{B}_0 y_k(t) \\ y_{k+1}(t) =& \hat{C}x_{k+1}(t) + \hat{D}u_{k+1}(t) + \hat{D}_0 y_k(t) \end{aligned} \tag{1}$$

Here on pass k, $x_k(t)$ is the $n \times 1$ state vector, $y_k(t)$ is the $m \times 1$ pass profile vector, and $u_k(t)$ is the $r \times 1$ vector of control inputs. To complete the process description, it is necessary to specify the boundary conditions, i. e. the state initial vector on each pass and the initial pass profile. Here no loss of generality arises from assuming these to be of the form $x_{k+1}(0) = d_{k+1}$, $k \ge 0$, and $y_0(t) = f(t)$, where d_{k+1} is an $n \times 1$ vector of known constant entries and $f(t)$ is an $m \times 1$ vector whose entries are known functions of t over $0 \le t \le \hat{\alpha}$. To obtain a time-delay representation (Dymkov *et al*, 2004) of processes described by (1) (for the case $1 \le k \le N$ where N is a fixed integer), introduce the new variables $x : [0, \hat{\alpha}N] \to \mathbb{R}^n$, $y : [0, \hat{\alpha}N] \to \mathbb{R}^m$, $u[0, \hat{\alpha}N] \to \mathbb{R}^r$, where

$$x(t) = \begin{cases} x_1(t), & 0 < t < \hat{\alpha}, \\ x_2(t-\hat{\alpha}), & \hat{\alpha} < t < 2\hat{\alpha}, \\ \cdots\cdots & \cdots\cdots\cdots \\ x_N(t-\hat{\alpha}(N-1)), & \hat{\alpha}(N-1) < t < \hat{\alpha}N \end{cases}$$

and $y(t)$ and $u(t)$ are defined in an identical manner. Then the repetitive process dynamics described by (1) are equivalently described by the following differential time-delay system

$$\begin{aligned} \begin{bmatrix} \frac{d}{dt} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \hat{A} & 0 \\ \hat{C} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\ + \begin{bmatrix} 0 & \hat{B}_0 \\ 0 & \hat{D}_0 \end{bmatrix} \begin{bmatrix} x(t-\hat{\alpha}) \\ y(t-\hat{\alpha}) \end{bmatrix} \\ + \begin{bmatrix} \hat{B} & 0 \\ 0 & \hat{D} \end{bmatrix} \begin{bmatrix} u(t) \\ u(t) \end{bmatrix} \end{aligned} \tag{2}$$

with boundary conditions

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}, \quad t \in [-\hat{\alpha}, 0]. \tag{3}$$

and I_m denotes the identity matrix in $\mathbb{R}^m$. In order to complete the correspondence between (2) and (1) we must impose additional constraints at $t = \hat{\alpha}k$, $k = 1, \ldots N-1$, which demand that the solution $x(t)$ is discontinuous. This leads to the so-called non-local conditions of the form

$$x(k\hat{\alpha}+) = d_k, \quad k = 1, \cdots, N-1, \tag{4}$$

where $x(k\hat{\alpha}+)$ denotes the value of $x(t)$ as $t \to k\hat{\alpha}$ from the right. We also assume that the control $(u(t))$ and pass profile $(y(t))$ vectors are continuous from the right-hand side at $t = \hat{\alpha}k$, $k = 1, \ldots, N-1$.

The system (2) can now be considered as a special case of the following pair of time delay differential and difference equations

$$\begin{aligned} \dot{x}(t) = \quad & Ax(t) + A_{-1}x(t-h) + B_0 y(t) + \\ & +B_{-1}y(t-h) + Bu(t) \end{aligned} \tag{5}$$

$$\begin{aligned} y(t) = & Cx(t) + C_{-1}x(t-h) + D_{-1}y(t-h) + \\ & +Du(t), \ t \in T = [0, \alpha] \end{aligned} \tag{6}$$

with boundary conditions

$$\begin{aligned} x(t) =& f(t), \ t \in [-h, 0), \ x(0) = x_0, \\ y(t) =& g(t), \ t \in [-h, 0] \end{aligned} \tag{7}$$

and $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^r$, where α, h are given real numbers such that $h < \alpha$. The repetitive process (2) and, hence, (1) follows immediately on choosing the matrices in (5)–(6) as

$$A = \hat{A}, \ A_{-1} = 0, \ B_0 = 0, \ B_{-1} = \hat{B}_0, \ B = \hat{B},$$

$$C = \hat{C}, \ C_{-1} = 0, \ D_{-1} = \hat{D}_0, \ D = \hat{D}.$$

and $\alpha = \hat{\alpha}N$, $h = \alpha$.

As a first step in analysis, we consider the case when the non-local conditions of (4) are absent. This can arise, for example, in the modeling of physical processes where the state initial vector for the current pass is equal to the state vector at the last evaluation on the previous pass, i.e. $x_{k+1}(0) = x_k(\alpha)$, that occur often in machining operations. If this condition holds, then the presence of non-local impulse initial conditions, and hence a source of significant analysis difficulties, can be avoided.

The pair of the functions $\big(x(t), y(t)\big)$ is termed a solution of the system (5)–(7) for a given control vector $u(t)$, if they satisfy the differential equation (5) almost everywhere on the interval $[0, \alpha]$ and the difference equation (6) for all $t \in [0, \alpha]$. Under the assumptions made here, it can be shown that the solution $x(t)$ is absolutely continuous and also that $y(t)$ is piecewise continuous on the interval $[0, \alpha]$.

The solution of the system (5)–(7) can be constructed using a step-by-step, or recurrent, procedure for each sub-interval of the form $[ih, (i+1)h)$, $i = 0, 1, \ldots, q_\alpha$, where $q_\alpha = \left[\frac{\alpha}{h}\right]$ denotes the integer part of the fraction $\frac{\alpha}{h}$. In particular, introduce the function $F(t,\tau)$ as a solution of the following differential equation

$$\frac{\partial F(t,\tau)}{\partial \tau} = -\sum_{j=1}^{q_t+1} F(t,\tau+(j-1)h)H_j,$$
$$F(t,\tau) \equiv 0,\ \forall\ \tau > t,\ F(t,t-0) = I_n.$$

Then it is straightforward to show that the standard recurrent procedure here leads to the following formula for the solutions of the system (5)–(7):

$$x(t) = s(t,f,g,x_0) + \int_0^t S(t,\tau)u(\tau)d\tau, \quad (8)$$

$$s(t,f,g,x_0) = \sum_{j=1}^{i+1}\int_{-h}^{0} F(t,\tau+(j-1)h)H_j f(\tau)d\tau$$
$$+\int_{-h}^{0} F(t,\tau+(i+1)h)\big[P_{i+1}f(\tau)+Q_{i+1}g(\tau)\big]d\tau$$
$$+F(t,0)x_0, \quad S(t,\tau) = \sum_{j=1}^{i+1} F(t,\tau+(j-1)h)V_j,$$

$$\begin{aligned} y(t) =& CF(t,0)x_0 + \int_{-h}^{0} CF(t,\tau)H_1 f(\tau)d\tau + \\ &+\int_{-h}^{0} CF(t,\tau+h)\Big[P_1 f(\tau)+Q_1 g(\tau)\Big]d\tau \\ &+\int_0^t CF(t,\tau)V_1 u(\tau)d\tau + C_{-1}f(t-h) + \\ &+D_{-1}g(t-h)+Du(t),\ t\in[0,h) \end{aligned}$$

$$\begin{aligned} y(t) =& CF(t,0)x_0 + \sum_{j=0}^{q_t-1} M_{j+1}F(t-(j+1)h,0)x_0 \\ &+\sum_{l=0}^{q_t-1}\sum_{j=0}^{q_t-l}\int_{-h}^{0} M_l F(t-lh,\tau+jh)H_{j+1}f(\tau)d\tau \\ &+\sum_{l=0}^{q_t}\int_{-h}^{0} M_l F(t-lh,\tau+(q_t+1-l)h) \\ &\Big[P_{q_t+1-l}f(\tau)+Q_{q_t+1-l}g(\tau)\Big]d\tau + \\ &+\int_0^t R(t,\tau)u(\tau)d\tau + W_{q_t-1}f(t-q_t h) \\ &+\sum_{j=0}^{q_t} G_j u(t-jh) + K_{q_t-1}g(t-q_t h), t\geq h, \end{aligned} \quad (9)$$

$$R(t,\tau) = \sum_{l=0}^{q_t-1}\sum_{j=0}^{q_t-l} M_l F(t-lh,\tau+jh)V_{j+1},$$
$$M_{j+1} = D_{-1}^{j}(C_{-1}+D_{-1}C),\ G_j = D_{-1}^{j}D,$$
$$M_0 = C, K_i = D_{-1}^{i}C_{-1}, W_i = D_{-1}^{i+1}, j = 0,1,\ldots$$

$$\begin{aligned} H_j =& \big(B_0 D_{-1}^{j-1} + B_{-1}D_{-1}^{j-2}\big)\big(C_{-1}+D_{-1}C\big), \\ H_2 =& A_{-1} + B_0\big(C_{-1}+D_{-1}C\big) + B_{-1}C, \\ H_1 =& A + B_0 C,\ j = 3,\ldots,q_t+1, \\ V_j =& \big(B_0 D_{-1}^{j-1} + B_{-1}D_{-1}^{j-2}\big)D, \\ V_1 =& B + B_0 D,\ j = 2,\ldots,\ q_t+1, \\ Q_i =& \big(B_0 D_{-1} + B_{-1}\big)D_{-1}^{i-1},\ P_1 = A_{-1} + B_0 C_{-1} \\ P_i =& \big(B_0 D_{-1} + B_{-1}D_{-1}^{i-2}\big)C_{-1} \end{aligned} \quad (10)$$

3. CONTROLLABILITY

In this section we introduce the notion of controllability with respect to initial function that play a significant role for applications area. Note here that there exists more than one distinct concept of controllability for repetitive processes/time delay systems, see, for example, the work of Kirillova *et al*, 1976; Manitius and Triggiani, 1978; Dymkov *et al*, 2003b. Also this area is far from being complete for linear repetitive processes, either in terms of the basic theory or its implications in terms of, for example, performance and the structure of control schemes. One of the feasible controllability notion was studied in (Dymkov *et al*, 2003a). It is so-called pass-profile controllability. Physical motivation for this form of controllability is the requirement that the pass profile vector take preassigned values at particular points along the pass. Formally, it is introduced as follows.

Definition 1. The system (5)–(7) is said to be pass profile controllable at the given points $\beta_0,\beta_1,\ldots,\beta_\nu$, such that $0=\beta_0<\beta_1<\ldots\beta_\nu\leq\alpha$, if for any $c_i\in\mathbb{R}^m,\ i=0,\ldots,\nu$ there exists a control vector $u(t),\ t\in[0,\alpha]$ such that the solution $y(t,g,f,x_0,u)$ of the system (5)–(7) corresponding to zero initial conditions $g(t)=0,\ t\in[-h,0),\ f(t)=0,\ t\in[-h,0),\ x_0=0$ satisfies the following conditions

$$y(\alpha-\beta_j,0,0,0,u) = c_j,\ j=0,1,\ldots,\nu \quad (11)$$

In general, for the differential systems with retarded arguments and the hybrid differential- difference systems, in particular, a key role plays the so-called pointwise completness and related phenomena of controllability with respect to initial data. In order to formulate this notion we introduce the following notations. Let $C^n[-h,0],\ h>0$ denotes the vector space of the continuous n-

vector function $f : [-h, 0] \to \mathbb{R}^n$. The solution of system (5)—(6) (in the absence of input actions, i. e. with $B = 0$, $D = 0$) corresponding to the initial data (7) where $f \in C^n[-h, 0]$, $g \in C^m[-h, 0]$, $x_0 \in \mathbb{R}^n$ denote by $x(t) = x(t, f, g, x_0)$, $y(t) = y(t, f, g, x_0)$. Reachability set for the state variable $x(t)$ of the system (5)—(7) at the given moment $t^* \in [0, T]$ is defined as follows

$$\begin{aligned} \mathcal{R}_x(t^*) = \{ & x \in \mathbb{R}^n : x = x(t^*, g, f, x_0), \\ & \forall\, f \in C^n[-h, 0],\ g \in C^m[-h, 0],\ x_0 \in \mathbb{R}^n \} \end{aligned} \tag{12}$$

By analogy the reachability set for the pass profile $y(t)$ of the system (5)—(7) at the given moment $t^* \in [0, T]$ is

$$\begin{aligned} \mathcal{R}_y(t^*) = \{ & y \in \mathbb{R}^m : y = y(t^*, g, f, x_0), \\ & \forall\, f \in C^n[-h, 0],\ g \in C^m[-h, 0],\ x_0 \in \mathbb{R}^n \} \end{aligned} \tag{13}$$

For many cases an essential question is: Can one reach the desired state and/or pass profile position by a proper choice of the initial data? The following definition is a formal description of this problem.

Definition 2. It is said that the system (5)—(7) is pointwise complete on the interval $[0, T]$ if

$$\mathcal{R}_x(t) = \mathbb{R}^n \ \text{ and } \mathcal{R}_y(t) = \mathbb{R}^m \ \forall\, t \in [0, T]. \tag{14}$$

If for some $t^* \in [0, T]$ the conditions (14) are not true then the system is called pointwise degenerate at the moment t^*.

The notion of pointwise completness was introduced first by Weiss, 1967 for studying of controllability of linear differential time delay systems. Some details and an overview of existing results can be found in survey by Metelskii, 1994. It is obviously that ordinary linear differential system of the form $\dot{x}(t) = Ax(t)$ are pointwise complete since for any t^* and $x^* \in \mathbb{R}^n$ there exists $x(0) = x_0 \in \mathbb{R}^n$ such that the corresponding solution satisfies the condition $x(t^*, x_0) = x^*$. Also, it is proved that each stationary linear differential system with constant time delay is pointwise complete in the case $n = 2$. The following example shows that the presence "difference" equation in the hybrid system destroys the pointwise completness of differential time delay system with $n = 2$.

3.1 Example

Consider the hybrid system of (5)—(7) on the interval $t \in [0, T]$ where $h \le T \le 2h$, $n = 2$, $m = 2$, $h = \ln 2$ and the following choice of the matrices

$$\begin{gathered} A = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, A_{-1} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, B_{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \\ B_0 = 0,\ D_{-1} = 0,\ B = 0,\ D = 0, \\ C = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix},\ C_{-1} = \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix} \end{gathered} \tag{15}$$

Substituting the function $y(t)$ from second equation into the first of the system (5)—(7) corresponding to the given choice of matrices leads to the following time delay system

$$\begin{aligned} \dot{x}(t) = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 & 0 \\ -1 & 2 \end{bmatrix} x(t-h) \\ + \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} x(t-2h) \end{aligned} \tag{16}$$

Thus the state variable of the considered hybrid system (15) is described by the retarded differential system (16) with multiple delays. For simplicity, denote next the matrices involved in (16) by A, A_1, A_2, respectively. It is known (see, Metelskii, 1994, for example) that the linear stationary differential system with multiple delays is pointwise complete if, and only if, the following conditions

$$rank M^0 = n + n_1, n_1 = \sum_{i=1}^{N} rank M_i(\lambda_i) \tag{17}$$

hold. Here the matrices M^0 and $M_i(\lambda_i), \lambda_i \in \mathbb{C}$ are defined by spectral parameters of the operator

$$W(\lambda, e^{-\lambda h}) = (\lambda I - A - e^{-\lambda h} A_1 - e^{-2\lambda h} A_2) \tag{18}$$

associated with the system (16). In the considered case we have

$$W(\lambda, e^{-\lambda h}) = \begin{bmatrix} \lambda + 2e^{-\lambda h} & -2 \\ e^{-\lambda h} + 2e^{-2\lambda h} & -\lambda - 1 - 2e^{-\lambda h} \end{bmatrix}$$

and $\det W(\lambda, e^{-\lambda h}) = \lambda^2 - \lambda$. Hence, the eigenvalues are $\lambda_1 = 0$ and $\lambda = 1$. Further, noting $h = \ln 2$, we have

$$M_1(\lambda_1) = W(\lambda, e^{-\lambda h})|_{\lambda=0} = \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix},$$

$$M_2(\lambda_2) = W(\lambda, e^{-\lambda h})|_{\lambda=1} = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$$

and the constant $(n+1)n \times n^2 = (6 \times 4)$-matrix M^0 is defined as follows

$$M^0 = \begin{bmatrix} M_1(\lambda_1) & O \\ O & M_2(\lambda_2) \\ I & I \end{bmatrix},\ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It is easy to verify that

$$rankM_1 = rank\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} = 1,$$

$$rankM_2 = rank\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} = 1$$

and

$$rankM^0 = rank\begin{bmatrix} 2 & -2 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = 3.$$

Hence

$$3 = rank\ M^0 < n + n_1 = 4.$$

and the conditions (17) show, therefore, that the considered system is not pointwise complete for the delay value $h = \ln 2$. Note that the eigenfuncions corresponding to the given eigenvalues $\lambda_1 = 0,\ \lambda_2 = 1$ are $\phi_1(t) = (1,1)^T,\ \phi_2(t) = (e^t, e^t)^T$. It is obviously that the rang of the fundamental matrix the entries of which are the given eigenfunctions is equal 1. Hence the linear space formed by these basic functions is isomorphic to space $\mathbb{R}$. This means again that the system under consideration is degenerate.

The given example shown clear the need for studying the cases when arbitrary state/pass profile position can be achieved by the proper choice of the initial data and control functions. One of a formal definition of this notion in the simplest case can be given as follows

Definition 3. The system (5)—(7) (with $B = 0,\ D = 0,\ x_0 = 0$) is called to be state controllable with respect to initial data at the given moment $t = T$ if for any n-vector $c_T \in \mathbb{R}^n$ there exist the initial functions $g(t),\ f(t),\ t \in [-h, 0]$ such that the corresponding solution $x(t, g, f, x_0)$ of the system (5)—(7) satisfies the following condition

$$x(T, g, f, x_0) = c_T \tag{19}$$

This notion has clear physical motivation connected with testing structural properties of the model under consideration. Then we have the following result.

Theorem 4. The system (5)—(7) (with $B = 0,\ D = 0$) is pass profile controllable with respect to initial data at the given moment $t = T$ if, and only if,

i) system (5)—(7) is pointwise complete;

ii)

$$rank\{H_0^i[H_1, H_2, \ldots, H_{qT}, Q_{gT}, P_{qT}],\quad i = 0, \ldots, n\} = n \tag{20}$$

4. OPTIMIZATION

Consider the following time optimal control problem for the process (5)—(7) announced first in the paper (Dymkou and Jank, 2004): For the given initial data $x(t) = f(t),\ t \in [-h, 0),\ x(0) = x_0,\ y(t) = g(t),\ t \in [-h, 0]$ find

$$T^0 = arg\min_u\{T\} \tag{21}$$

over the solutions of the process (5)—(7) subject to state constraints

$$x(t) \equiv 0,\ t \in [T - h, T] \tag{22}$$

and integral control constraints

$$U_T(\cdot) \doteq \left\{u(\cdot) : \int_0^T u^T(\tau)u(\tau)d\tau \le 1\right\} \tag{23}$$

In effect, the solution of this problem will drive the system dynamics subject to control constraint to the zero equilibrium state as fast as possible. Also let T be a fixed time. Then the class of admissible control vectors $u(t),\ t \in [0, T]$ is the set of the all piecewise continuous vectors such that $u(t) \in U,\ t \in [0, T]$, where U is a compact convex set in $\mathbb{R}^r$. The following result is hold.

Theorem 5. For the given initial data $f(t), g(t),\ t \in [-h, 0), x(0) = x_0$ the moment T^0 is optimal if, and only if, T^0 is a minimal root for the inequality

$$\Lambda(T) \le 0 \tag{24}$$

where

$$\Lambda(T) =: \max_{\|g\|=1}\left\{g^T s(T - h, f, g, x_0) + \inf_{u \in U_T(\cdot)} \int_0^{T-h} g^T S(T - h, \tau)u(\tau)d\tau\right\} \le 0 \tag{25}$$

We shown that for the considered case the optimal control function can be presented in the flexible form as follows.

Theorem 6. Optimal time T^0 in optimization problem (21)—(23) is the minimal root of the inequality

$$\max_{\|g\|=1}\left\{g^T s(T - h, f, g, x_0) + \int_0^{T-h} g^T S(T - h, \tau)u_g(\tau)d\tau\right\} \le 0 \tag{26}$$

and the corresponding optimal control is

$$u^0(t) = \begin{cases} u_{g^0}(t), \ t \in [0, T^0 - h) \\ Nx^0(t-h) + My^0(t-h), \\ t \in [T^0 - h, T^0] \end{cases} \qquad (27)$$

where the vector g^0 realizes maximum in (26), $u_g(t) = u_1(t) + u_2(t)$, $u_2(t) = \frac{1}{\lambda} L(t) g$ and

$$N = \left[E + DB^{-1}B_0\right]^{-1}\left[C_{-1} - DB^{-1}A_{-1}\right],$$

$$M = \quad \left[E + DB^{-1}B_0\right]^{-1}\left[D_{-1} - DB^2_{-1}\right].$$

Moreover, the vector $u_1(t)$ and the $(n \times r)$- matrix $L(t)$ satisfy the following integral equations

$$\int\limits_0^{T-h} K(\theta, t) u_1(\theta) d\theta = -2u_1(t)(I + \mathcal{G}(t)) - \psi(t) - \varphi(t)$$

and

$$2L(t) + S(T-h, t) + \int\limits_0^{T-h} K(\theta, t) L(\theta) d\theta = 0.$$

5. CONCLUSION

In this paper differential repetitive processes are studied from the perspective of differential delayed systems. The new mathematical models for this class of systems have been introduced and primary analysis is provided. First of all, the controllability and time optimal control have been outlined. It is necessary to add that this note covers only first attempts to investigate the differential repetitive processes from that point of view, and hence a rich material to be the subject for further work still remains. For example, new controllability and observability notions are of a significant interest for further investigations. In particular, another controllability notion includes so-called functional controllability (see, for example, Manitius and Triggiani, 1978) when is required to drive the state variables at the final interval $[\alpha, \alpha + h]$ to the pre-assigned functions $x(t) = \varphi(t), y(t) = \psi(t), t \in [\alpha, \alpha + h]$. The developed results for the linear process (5)—(7) can be applied also to obtain the necessary optimality conditions for optimal control of nonlinear models, too. These problems are subject of ongoing work and will be reported in due course.

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