Control problems for discrete Volterra systems

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Abstract

In this paper a rigorous Volterra control system theory is developed. In framework of the uniformed view based on operator approach the main control problems such as stability, stabilizability, controllability, linear-quadratic optimization and feedback control problems are considered for linear discrete Volterra equations. It is shown that Volterra operator techniques play a key role to study the main structural properties of the system under consideration. Using the presentation of this operator in the ring of power series allows us to apply some algebraic methods for research.

1 Basic notation and definitions

Let E be a finite dimensional normed space over the complex field C with norm $|\cdot|_E$, and Z_+ be the set of nonnegative integers. Denote by $s(Z_+, E)$ the linear space of all sequences on E, i.e. the functions $f : Z_+ \to E$. Let $b(Z_+, E)$ be the subspace from $s(Z_+, E)$ of all bounded functions, i.e. the functions $f : Z_+ \to E$ such that $\sup\limits_{k\in Z_+} |f(k)|_E < +\infty$. As known the space $b(Z_+, E)$ is dense in the space $s(Z_+, E)$ with respect to the topology of uniform convergence over the finite sets. The space $b(Z_+, E)$ can be converted to the normed space if the norm is given, for example, as $\|f\| = \sup\limits_{k\in Z_+} |f(k)|_E$. Moreover, it is Banach space.

Now let V be finite dimensional normed spaces over complex field C, $A_t \in \mathcal{L}(E, E), t \in Z_+, B \in \mathcal{L}(V, E)$, where $\mathcal{L}(E, V)$ denotes the Banach space of all linear operators from E to V.

The relation

$$x(t+1) = \sum_{i=0}^{t} A_i x(t-i) + Bu(t), t \in Z_+ \qquad (1)$$

will be called the linear discrete Volterra equation for the unknown sequence $x(t)$ with values in E. The function $u : Z_+ \to V$ is called usually as control input, the function $x : Z_+ \to E$ represents the state vector.

Definition 1. *For the given control function u we say that the function $x : Z_+ \to E$ is the solution of equation (1) if it satisfies to (1) for all $t \in Z_+$.*

It is clear that the equation (1) is solvable and any solution of this equation is defined by the initial condition $x_0 = x(0)$, and for any $x_0 \in E$ there is a unique solution $x(t)$ such that $x_0 = x(0)$. To emphasize the dependence of the sequence $x(t)$ on the initial data x_0 we write sometimes such a sequence as $x(t, x_0)$.

Define the set of linear operators $Q_t : E \to E,\ t \in Z_+$ by

$$Q_0 = I,\ Q_{t+1} = \sum_{i=0}^{t} A_i Q_{t-i}, \quad t \in Z_+, \qquad (2)$$

where I is the identity operator in E.

Then any solution of equation (1) with the initial condition x_0 can be written as follows

$$x(t, x_0) = \sum_{i=0}^{t-1} Q_i Bu(t-i-1) + Q_t x_0. \qquad (3)$$

Here it is supposed by definition that $x(0, x_0) = x_0$.

Some structural properties of (1) (for example, boundedness, stability, controllability and etc.) essentially depend on so-called Volterra operator $\mathcal{V} : s(Z_+, E) \to s(Z_+, E)$ defined as follows

$$(\mathcal{V}\varphi)(t) = \sum_{i=0}^{t} A_i \varphi(t-i),\ t \in Z_+. \qquad (4)$$

Due this matter we establish below some facts concerning to this operator.

2 Discrete Volterra operator

Assume that there are some fixed bases in E and V. Therefore, the linear operators A_i can be understood as matrices on the complex field C. Associate with each Volterra operator $\mathcal{V}$ its representation $\mathcal{V}(z)$ in the ring of power series defined by

$$\mathcal{V}(z) = \sum_{i=0}^{\infty} A_i z^i, \quad z \in C. \qquad (5)$$

It is obvious that the matrix function $\mathcal{V}(z)$ is a linear map $E \rightarrow V$ for each z from the unit disk $\mathcal{D} = \{z \in C : |z| \leq 1\}$. Now, suppose that the matrices A_i are such that the power series (5) converge in some domain $\mathcal{G}$ containing the unit disk $\mathcal{D}$. Under given assumptions, $\mathcal{V}$ is a linear bounded operator.

Lemma 1. *If $E = V$ and $\det \mathcal{V}(z) \neq 0$ for $|z| \leq 1$, $z \in C$, then Volterra operator $\mathcal{V}$ is invertible.*

As known, the matrix $\mathcal{V}(z)$ can be transformed by using elementary transformation to the simple form

$$\mathcal{V}(z) = \sigma_1(z)p(z)\sigma_2(z), z \in \mathcal{G}, \tag{6}$$

where $\sigma_1(z)$ and $\sigma_2(z)$ are square matrices of appropriate dimension that are analytical in $\mathcal{G}$ and have nonzero determinants at all points of the closed unit disk $\mathcal{D}$; the matrix $p(z)$ has the same dimensions as $V(z)$ and all elements of $p(z)$ are zero except possibly for those on the diagonal starting from the upper left-hand corner which are polynomials with unit leading coefficient and with all roots in the closed unit disk $\mathcal{D}$. Without loss of generality one can assume that nonzero diagonal elements $p_1(z), ..., p_l(z)$ of matrix $p(z)$ are situated in the first l rows of $p(z)$. In other words, the matrix $p(z)$ can be written in the form

$$p(z) = \begin{pmatrix} p_1(z) & 0 & \dots & \dots & \dots & 0 \\ 0 & p_2(z) & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & \dots & p_l(z) & \dots & 0 \\ & \dots & \dots & \dots & \dots & \\ 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix}.$$

Moreover, each nonzero polynomial $p_{j+1}(z)$ can be divided by $p_j(z)$. In what follows it is shown that Volterra operators Q_1 and Q generated by the matrices $\sigma_1(z)$ and $\sigma_2(z)$, respectively, are invertible.

Lemma 2. *The Volterra operator $\mathcal{V} : b(Z_+, E) \rightarrow b(Z_+, E)$ is surjective if, and only if, $rank\mathcal{V}(z) = n$ $(n = dim\ E)$ for all $z \in C, |z| \leq 1$.*

Proof. *Sufficiency.* On the contrary, let rank $V(z_0) < \dim E$. Assume that the representation $V(z)$ for the operator V is factored in the form $V(z) = \sigma_1(z)p(z)\sigma_2(z)$. Without loss of generality we suppose that $p(z) = l(z)p_1(z)$, where $p_1(z)$ is the matrix with the same properties as $p(z)$, and

$$l(z) = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ & & & \ddots \\ 0 & & & z - z_0 \end{pmatrix}.$$

Due this presentation the operator Volterra corresponding to the matrix $p(z)$ can be given by the following composition $P = LP_1$. Now prove that the operator $L : b(Z_+, E) \rightarrow b(Z_+, E)$ is not surjective . It follows that the operator V is not surjective, also. Write the system $(Lf)(s) = \beta(s)$ in coordinate form and consider its n-th equation which in accordance with the presentation of $l(z)$ is

$$-z_0 f_n(s+1) + f_n(s) = \beta(s),\ s \in Z_+. \tag{7}$$

Then, for $z_0 \neq 0$ we obtain $f_n(s) = -z_0^{-(s+1)} \sum_{i=0}^{s} \beta_n(i) z_0^i$. Let $|z_0| = 1$. Put $\beta_n(s) = c z_0^{-s}$, where c is an arbitrary nonzero number. It is obvious that the sequence $\{\beta_n(s)\}$ is bounded. Hence

$$\sup_s |f_n(s)| = \sup_s |\sum_{i=0}^{s} \beta_n(i) z_0^i| = \sup_s |(s+1)c| = \infty.$$

It means that for the chosen $\beta(s)$ the considered equation has not any solution in the space $b(Z_+, E)$. It is proved nonserjectivity of L in the case $|z_0| = 1$. If $0 < |z_0| < 1$, then we set $\beta_n(0) = 1, \beta_n(s) = 0,\ s > 0$ in (10). Hence $f_n(s) = -z_0^{-s}$, that is $\sup_s |f_n(s)| = \infty$. In the case $z_0 = 0$, it is evidently that the equation (7) has not any solution for $\beta_n(0) \neq 0$. Thus, the operator V is not surjective that is contrary to the Lemma.

Lemma 3. *The Volterra operator $\mathcal{V} : b(Z_+, E) \rightarrow b(Z_+, E)$ is injective if, and only if, $rank\mathcal{V}(z) = n$ $(n = dim\ E)$ for some $z \in C, |z| \leq 1$.*

Proof. If the mapping V is injective, then the equation $V(z)x(z) = 0$ or the equivalent system of equations $p(z)x(z) = 0$ has only the zero solution. Let the operator V is injective and, on the contrary, rank $V(z) < n$ for all $z \in C$, $|z| \leq 1$. In this case the last column of the matrix $p(z)$ is equal zero. Therefore $x(z) = (0, ..., 0, 1)$ is a nonzero solution to the equation $p(z)x(z) = 0$. It is contrary to the preposition of the Lemma.

Conversely, let $V(z_0) = n$ for some $|z_0| \leq 1$. That is the matrix $p(z)$ can be presented in the form

$$p(z) = \begin{pmatrix} p_1(z) & & 0 \\ & \ddots & \\ 0 & & p_n(z) \end{pmatrix}, \tag{8}$$

where the all polynomials $p_i(z)$ are nonzero and its roots belong to the unit disk $\mathcal{D}$. Let Σ_1, P, Σ_2 be the operators Volterra corresponding to the presentation $\sigma_1(z)$, $p(z)$, $\sigma_2(z)$. It is obvious that the operators Σ_i, $i = 1, 2$ are invertible. We show that P is an injective operator. Consider the following system of equations

$$p_i(z)x_i(z) = 0, \quad z \in C, \quad i = 1, 2, ..., n. \tag{9}$$

Since the polynomials $p_i(z)$ can be only a finite number of roots in the domain $\mathcal{D}$ then the given equation has only the zero solution $x_i(z) = 0$ in the ring of analytical in $\mathcal{D}$ functions. Hence the operator composition of the form $V = \Sigma_1 P \Sigma_2$ is an injective operator. Lemma is proved.

From Lemmas 1 and 2 follows immediately.

Corollary 1. *The Volterra operator $\mathcal{V} : b(Z_+, E) \rightarrow b(Z_+, E)$ is bijective if, and only if, $rank\mathcal{V}(z) = n$ $(n = dim\ E)$ for all $z \in C, |z| \leq 1$.*

Now the spectrum of the Volterra operator may be characterized as follows.

Lemma 4. *The spectrum $\Sigma(V)$ of the operator V can be evaluated by the formula*

$$\Sigma(V)=\bigcup_{|z|\le 1}\sigma(V(z)), \tag{1)0}$$

where $\sigma(V(z))$ denotes eigenvalues of the matrix $V(z)$.

These results may be applied successfully to the difference equations analysis. Let $A_i : E \to V$, $B_i : W \to E$, $i \in Z_+$ be linear operators, such that the power series $V(z)=\sum_{i\in Z_+} A_i z^i$, $B(z)=\sum_{i\in Z_+} B_i z^i$ are convergent in the unit disk $\mathcal{D}$.

Consider the following system of equations

$$\sum_{i=0}^{s} A_i x(s-i)+B_s y=\beta_s \quad \text{for all} \quad s\in Z_+ \tag{11}$$

with respect to unknown $x\in b(Z_+,E)$, $y\in W$ under given $\beta\in b(Z_+,E$ such as that the power series$\beta(z)=\sum_{i\in Z_+}\beta_i z^i$ is convergent in some domain containing the unit disk $\mathcal{D}$.

The system (11) is equivalent to the following equation for representations in the ring of power series

$$V(z)x(z)+B(z)y=\beta(z), \quad z\in\mathcal{D}. \tag{12}$$

Lemma 5. *Let $\dim E=1$. Then for any $B(z)$ there exists $\beta^*\in b(Z_+,E)$ such that the equation*

$$(z-z_0)x(z)+B(z)y=\beta(z), \quad |z_0|=1 \tag{13}$$

with respect to unknown $x\in b(Z_+,E)$, $y\in W$ has no solution for $\beta=\beta^*$.

Let $V(z)$ be analytical $(n_1 \times n_2)$-matrix $(n_1=\dim E, n_2=\dim V)$. The row $a(z)=(a_1(z),a_2(z),...,a_{n_1}(z))$, whose coordinates $a_i(z)$ are analytical functions at $z=z_0$ is called k-annihilating $(k\ge 1)$ at $z=z_0$ for matrix $V(z)$ if the n_2-vector function $(z-z_0)^{-k}a(z)V(z)$ is analytical at $z=z_0$.

Denote by $H^k_{z_0}[V]$ the set of all k-annihilating rows for $V(z)$. It is easy to verify that if $a_1(z)\in H^k_{z_0}[V], a_1(z)\in H^k_{z_0}[V]$ then its sum $a_1(z)+a_2(z)$ belongs to $H^k_{z_0}[V]$ also. Moreover, if the function $\lambda(z)$ is analytic at $z=z_0$ then $\lambda(z)a_1(z)\in H^k_{z_0}[V]$, too. Thus, the set $H^k_{z_0}[V]$ is a modulus on the ring of all analytical functions at $z=z_0$. Let the matrix $V(z)$ is factored in the form (8). Denote by ρ_i the multiplicity of the root $z=z_0$ of the polynomial $p_i(z)$ in (9). It is obvious that $\rho_1\le\rho_2\le...\le\rho_j$. This chain may be extended by putting $\rho_0=0$, $\rho_{j+1}=\rho_{j+2}=...=\infty$. Then for each integer k there exists an index i such that $\rho_i\le k<\rho_{i+1}$. It is known that the rows

$$\sigma_{11}^{-1}(z)(z-z_0)^{k-\rho_1},...,$$

$$\sigma_{1i}^{-1}(z)(z-z_0)^{k-\rho_i},\sigma_{1i+1}^{-1}(z),...,\sigma_{1n_2}^{-1}(z)$$

form a basis for modulus $H^k_{z_0}[V]$. Here $\sigma_{1j}^{-1}(z)$ denotes j-th row for invert matrix $\sigma_1^{-1}(z)$.

Let $B(z)$ be $(n_1\times n_3)$-matrix analytical at $z=z_0$ $(n_3=\dim W)$. Denote by $\mathfrak{L}_{z_0}[V,B]$ the linear span of all n_2-dimensional rows of the form $[a(z)B(z)]^{(i)}_{z=z_0}$, $i=0,1,...,k-1$, where $a(z)\in H^k_{z_0}[V]$ and the index k takes value 1,2,... . Here, $[f(z)]^{(i)}_{z=z_0}=\frac{d^i f(z_0)}{dz^i}$ is i-th derivative at the point $z=z_0$.

Remark 1. The algebraic structure of the linear space $\mathfrak{L}_{z_0}[V,B]$ can be characterized in the following way. In each module $H^k_{z_0}[V]$ we take a basis $h^k_1(z),...,h^k_r(z)$. It was shown that the linear span of the n_3- dimensional rows

$$[h^k_1(z)B(z)]^{(k-1)}_{z=z_0},...,[h^k_r(z)B(z)]^{(k-1)}_{z=z_0}, k=1,2,...$$

coincides with $\mathfrak{L}_{z_0}[V,B]$. Moreover, if $n_1\ge n_2$ and $j=n_2$ in (9), then $\mathfrak{L}_{z_0}[V,B]$ coincides with the linear span of the rows

$$[\sigma_{1i}^{-1}(z)B(z)]^{(k)}_{z=z_0}, \quad i=1,...,n_2; k=0,...,\rho_i-1. \tag{14}$$

Denote $R_V=\{z\in C, |z|<1, \text{rank}V(z)<n_2\}$. The set R_V either coincides with $\mathcal{D}$ (when $n_1=\dim E_1<\dim E_2=n_2$ or $n_1\ge n_2$ and $p_{n_2}\equiv 0$ in (9)), or otherwise it contains a finite number of elements $z_1,...,z_r$, which are the roots of equation $p_{n_2}(z)=0$. We denote the multiplicity of a root z_i of a polynomial $p_j(z)$ from (9) by h_{ij} ($h_{ij}=0$ when $p_j(z_i)\ne 0$) and put $\rho(V)=\sum_{j=0}^{n_2}\sum_{i=0}^{r} h_{ij}$. As known, the number $\rho(V)$ is independent on transformations which reduce the matrix $V(z)$ to the factorization (8). It is called also the singularity power of $V(z)$ in the unit disk $\mathcal{D}$.

Theorem 1. *Equation (11) solvable in class $b(Z_+,E)$ for any $\beta\in b(Z_+,V)$ if and only if*

1) $\text{rank}V(z)=n_2 \quad (n_2=\dim V) \quad \text{for all} \quad |z|=1;$

2) $\dim\{\mathfrak{L}_{z_1}[V,B]+...+\mathfrak{L}_{z_r}[V,B]\}=\rho(V),$

$z_i\in R_V,\ i=1,2,...,r.$

Proof. *Necessity.* Assume that the matrix $V(z)$ is factored to the form of (8). Then equation (11) is equivalent to (12) which is also equivalent to the following equation

$$p(z)\sigma_2(z)x(z)+\sigma_1^{-1}(z)B(z)y=\sigma_1^{-1}(z)\beta(z). \tag{15}$$

Since $\sigma_1(z)$ and $\sigma_2(z)$ are invertible in $\mathcal{D}$, then solvability of (15) for any $\beta(z)$ is equivalent to solvability for any $\beta(z)$ of the equation

$$p(z)x(z)+\sigma_1^{-1}(z)B(z)y=\beta(z). \tag{16}$$

Suppose, hence, that equation (11) is solvable. Now, our goal is to prove that conditions 1) and 2) are true. First, consider the condition 1). On the contrary, let there exists $z_0\in C$, $|z_0|=1$ such that $\text{rank}V(z_0)<n_2$. At the first stage, suppose that $n_2\le n_1$. For this case, the polynomial p_{n_2} in the matrix (9) can be presented as $p_{n_2}(z)=(z-z_0)g(z)$.

Equation (16) may be rewritten in a coordinate form and its last equation is

$$(z-z_0)g(z)x_{n_2}(z)+[\sigma_1^{-1}(z)B(z)y]_{n_2}=\beta_{n_2}(z). \quad (17)$$

It is easy to see from Lemma 3 that the equation (17) is not solvable for arbitrary $\beta_{n_2}(z)$. This contradiction proves the condition 1) for the case $n_2 \leq n_1$.

Let now $n_2 > n_1$. For this case, the last row in the matrix $p(z)$ in (9) is zero. Hence, the equation (17) may be rewritten in the form

$$[\sigma_1^{-1}(z)B(z)y]_{n_2}=\beta_{n_2}(z). \quad (18)$$

Since y belongs to the finite-dimensional space V, then the infinite-dimensional equation (18) cannot be solved for an arbitrary right-hand side $\beta(z)$. Thus, the condition 1) is proved for this case too.

Consider now the condition 2). Since $\mathrm{rank}V(z) = n_2$, $|z|=1$, then the index l in the matrix $p(z)$ satisfies $l=n_2$, so there are no the lower diagonal zeros. Assume, next, that the polynomials $p_i(z)$, $i=1,2,...,n_2$ are given as

$$p_i(z)=(z-z_1)^{\rho_1^i}\cdot\ldots\cdot(z-z_r)^{\rho_r^i}, \quad i=1,2,...,n_2.$$

Equation (16) for this case may be rewritten in a coordinate form as

$$p_1(z)x_1(z)+\sigma_{11}^{-1}(z)B(z)y=\beta_1(z),$$

....

$$p_{n_2}(z)x_{n_2}(z)+\sigma_{1n_2}^{-1}(z)B(z)y=\beta_{n_2}(z). \quad (19)$$

Since, in accordance with the preposition, equations (19) are solvable for arbitrary $\beta(z)$, then the following system of equations

$$\left[\sigma_{1i}^{-1}(z)B(z)\right]_{z=z_j}^{(l)} y=[\beta_i(z)]_{z=z_j}^{(l)}, \quad (20)$$

$$i=1,2,...,n_2;\ j=1,2,...,r;\ l=0,1,...,\rho_j^i-1,$$

is solvable with respect to $y \in W$ for any $\beta(z)$.

Since $\beta(z)$ is an arbitrary function, then we may assume that the right-side of (20) are arbitrary, too. Therefore, the rank of the matrix formed by coefficients of the left-side in (20) is equal to the number of equations in (20). But according to Remark 1 the rows of this matrix form a basis for the linear space

$$\mathfrak{L}_{z_1}[V,B]+\ldots+\mathfrak{L}_{z_k}[V,B].$$

Since the number of equations in (20) is equal $\rho(V)$, then

$$\mathrm{rank}\{\mathfrak{L}_{z_1}[V,B]+\ldots+\mathfrak{L}_{z_k}[V,B]\}=\rho(V).$$

Sufficiency. Assume that the conditions 1) and 2) of Lemma are satisfied. Then for any $\beta(z)$ there exists the solution $y \in W$ to the system of equations (20). Therefore, the system of equations (19) is solvable too. It indicates that equation (15) is satisfied whence it follows that equation (11) is solvable.

3 Boundedness, stability and stabilization

The obtained results we use now for investigation of structural properties of (1). At the first we assume that $u \in b(Z_+, V)$.

Theorem 2. *If* $L=\sum_{t\in Z_+}|A_t|<1$, *then all solutions of equation (1) are bounded on* Z_+ *for any* $u \in b(Z_+,V)$. *Moreover if* $\sum_{t\in Z_+}|u_t|<\infty$ *then* $\sum_{t\in Z_+}|x_t|<\infty$.

Stability problem for some classes of discrete Volterra equation has been considered by [6 - 8], where the method of Lyapunov functions is used. We use algebraic method based on representation of Volterra operator in the ring of power series.

Consider the homogeneous Volterra equation

$$x(t+1)=\sum_{i=0}^{t}A_i x(t-i), \quad t \in Z_+. \quad (21)$$

Definition 2. *The system (21) is said to be exponentially stable if there exists a real number* q, $0<q<1$, *such that the inequality*

$$|x(x_0,t)|_E \leq \lambda |x_0| q^t$$

occur for each initial condition $x(t_0)=x_0$, *where* λ *is some positive real number.*

Theorem 3. *If the system (21) is exponentially stable then the following condition*

$$\det(\mathcal{V}(z)-\lambda I)\neq 0 \quad for\ all \quad |z|=1,\ |\lambda|\geq 1. \quad (22)$$

holds.

Proof. It is obvious that if (22) is true then $\det(\mathcal{V}(z)-\lambda I)\neq 0$ for $|z|\leq 1$, $|\lambda|\geq 1$. Therefore $\det(I-\mu\mathcal{V}(z))\neq 0$ for $|\mu|\leq 1$, $|z|\leq 1$. Since $\det(I-\mu\mathcal{V}(z))$ is analytical then there exists the number $\rho>1$ such that $\det(I-\mu\mathcal{V}(z))\neq 0$ for $|z|<\rho$, $|\mu|<\rho$. Hence

$$(I-\mu\mathcal{V}(z))^{-1}=\sum_{k=0}^{\infty}\mu^k\mathcal{V}^k(z)=\sum_{k=0}^{\infty}\mu^k\left(\sum_{i=0}^{\infty}A_i z^i\right)^k$$

$$=\sum_{k=0}^{\infty}\mu^k\sum_{i=0}^{\infty}A_i^{(k)}z^i, \quad |\mu|<\rho,\ |z|<\rho,$$

where the matrices $A_i^{(k)}$ is determined by ordinary manner. Put $\mu_0=\frac{\rho+1}{2}$, $z_0=\frac{\rho+1}{2}$. Here the above power series is convergent at the point (z_0,μ_0). Therefore there exists the constant $L>0$ such that $|\mu_0^k A_i^{(k)} z_0^i|\leq L$. It follows that $\left|A_i^{(k)}\right|\leq L(\frac{2}{\rho+1})^{i+k}$ for all $i,\ k\in Z_+$. In accordance with above the presentation of solutions to is given by the formula $X(z)=(I-z\mathcal{V}(z))^{-1}x_0$. Moreover, the value $x(s)$ for solution at the moment $t=s$ is equal to the coefficient at z^s in the power series expansion of $X(z)$. It is convenient the search of such coefficient at z^s for power series denote by δ_s. Hence

$$|x(s)|_E = |\delta_s(X(z))| = \left|\delta_s\left(\sum_{k=0}^{\infty} z^k \sum_{i=0}^{\infty} A_i^{(k)} z^i\right) x_0\right|$$

$$= \left|\left(A_{s-1}^{(1)} + A_{s-2}^{(2)} + ... + A_0^{(s-1)}\right) x_0\right| \le$$

$$\le (Lr^s + ... + Lr^s)|x_0| = sr^s L|x_0|,$$

where r denotes $\frac{2}{\rho+1}$. Put $q = \sqrt{r}$, $\lambda = \sup_s(Lsq^s)$. It is obvious that $r < 1$. Finally $|x(s)|_E \le \lambda|x_0|q^s$, where $0 < q < 1$. This completes the proof.

Definition 3. *Equation (21) is said to be stable,if for any* $\varepsilon > 0$ *there is a number* $\delta > 0$ *such that* $\|x(t, x_0)\| < \varepsilon$ *for any* $\|x_0\| < \delta$ *and all* $t \in Z_+$.

Definition 4. *Equation (21) is said to be asymptotically stable,if* $x(t, x_0) \to \infty$ *at* $t \to \infty$ *for any* $x_0 \in R^n$.

Define $\alpha = \sum_{i=0}^{\infty} \|A_i\|$.

Theorem 4. *If* $\alpha \le 1$ *then equation (1) is stable and it is asymptotically stable if* $\alpha < 1$.

Consider the discrete control Volterra equation (1), where the control low $u(t) \in V$, $t \in Z_+$ is given by

$$u(t) = \sum_{i=0}^{t} B_i x(t-i), \quad t \in Z_+. \tag{23}$$

Definition 5. *Equation (1) is said to be stabilizable by control (23) if there are operators* $B_0, B_1, ..., B_t, ...$ *from* V *to* E *such that the closed system*

$$x(t+1) = \sum_{i=0}^{t} (A_i + BB_i)x(t-i), \quad t \in Z_+,$$

is asymptotically stable.

Theorem 5. *Equation (1) is stabilizable by control (23) if the inequality* $\|A_i + BX\| \le \alpha_i$ *is solvable for any* $i \in Z_+$, *where* α_i *are nonnegative numbers such that* $\sum_{i=0}^{\infty} \alpha_i < 1$.

4 Controllability

Definition 6. *The control system (1) is said to be null-controllable in the class* $b(z_+, E)$ *if for any initial condition* $x(0) = x_0 \in E$ *there are* $j_0 \in Z_+$ *and control* $u(t)$, $0 \le t \le j_0 - 1$ *such that the corresponding solution* $x(t, x_0, u)$ *of (1) is bounded and the condition* $x(x_0, u, j_0) = 0$ *holds.*

Definition 7. *The control system (1) is said to be complete controllable in the class* $b(Z_+, E)$ *if there is* $j_0 \in Z_+$ *such that for any* $x_0 \in E$, $\beta \in b(Z_+, E)$, $\sum_{t=0}^{\infty} |\beta(t)| < \infty$ *there is control function* $u \in b(Z_+, E)$ *such that the corresponding solution* $x(t, x_0, u)$ *of (1) is bounded and the conditions* $x(x_0, u, 0) = x_0$, $x(x_0, u, t) = \beta(t)$, $t \ge j_0$ *hold.*

Theorem 6. *Volterra equation (1) is null-controllable iff* $\operatorname{rank}\{Q_t B, t = 0, ..., n-1\} = n$.

The representation of solution for (1) is given by

$$X(z) = zQ(z)BU(z) + Q(z)x_0, \quad z \in \mathcal{D} \tag{24}$$

where $Q(z) = \sum_{i=0}^{\infty} z^i Q_i$, $U(z) = \sum_{i=0}^{\infty} z^i u(i)$.

In accordance with Section 2 denote

$$R_T = \{z \in C, |z| < 1, \quad \operatorname{rank} zQ(z)B) < n\}.$$

Theorem 7. *If spectrum* $\Sigma(\mathcal{V})$ *of Volterra operator belongs to the interior of the unit disk* $\mathcal{D}$ *then the equation (1) is complete controllable iff*

1) $\operatorname{rank}\{T(z)\} = n$ *for all* $|z| = 1$;

2) $\dim\{\mathcal{L}_{z_1}[T, Q] + ... + \mathcal{L}_{z_r}[T, Q]\} = \rho(T)$, *where* $z_i \in R_T$, $i = 1, 2, ...r$; $\rho(T)$ *denotes the singularity power of the matrix* $T(z) = zQ(z)B$ *on the disk* $\mathcal{D}$; *the linear spaces* $\mathcal{L}_{z_i}[T, Q]$ *are constructed by special manner with help of matrices* $T(z)$ *and* $Q(z)$.

5 Optimal control

Consider now the discrete control system (1) where it is assumed that the admissible control function $u : Z_+ \to V$ belongs to the space l^2 of square summarized sequences of elements from V.

We consider quadratic cost functional of the form

$$J(u) = \sum_{t \in Z_+} [(Gx(t), x(t)) + (Ru(t), u(t))]. \tag{25}$$

Here G, R are selfadjoint operators, such that $Q \ge 0$, $R > 0$ (i.e. $(Qa, a) \ge 0$ for any $a \in E$, $(Rb, b) > 0$ for any $0 \ne b \in V$).

The optimization problem is to determine the admissible control such that the functional (25) is minimized and to establish the feedback form for the optimal control.

Theorem 8. *Let* $\sum_{i=0}^{\infty} \|A_i\| < 1$. *Then the extremal problem (1), (25) has a unique optimal solution.*

Proof. According to (3) the solution of the equation (1) can be written in the form

$$x(t) = \sum_{i=0}^{t-1} Q_i Bu(t-i-1) + Q_t x_0, \quad t \in Z_+.$$

Define now the mapping L in the space $l^2(V)$ by $(Lu)(t) = \sum_{i=0}^{t-1} Q_i Bu(t-i-1)$, $(Lu)(0) = 0$, $t \in Z_+$. It is clear that the set of values of the operator L belongs to the subspace $l_0^2(E)$ of sequences in E with zero first elements. By using this mapping the solution $x(t)$ corresponding to the admissible control $u(t)$ can be represented as follows $x = Lu + \omega$, where $\omega = (Q_0 x_0, Q_1 x_0, ...) \in l^2(E)$. In which case the considered functional $J(u)$ can be written as

$$J(u) = ((\mathcal{R} + L^* \mathcal{G} L)u, u) + 2(L^* \mathcal{G}\omega, u) + (\mathcal{G}\omega, \omega),$$

where $\mathcal{R} : l^2(V) \to l^2(V)$ and $\mathcal{G} : l^2(E) \to l^2(E)$ are defined by $(\mathcal{R}u)(t) = Ru(t), (\mathcal{G}x)(t) = Gx(t), t \in Z_+$. Since $\mathcal{R} > 0$, $\mathcal{G} \geq 0$ then $\mathcal{R} + L^*\mathcal{G}L$ is an invertible operator. Define

$$u^0 = -(R + L^* \mathcal{G} L)^{-1} L^* \mathcal{G}\omega. \tag{26}$$

Then

$$J(u) - J(u^0) = ((\mathcal{R} + L^* \mathcal{G} L)(u - u^0),\ (u - u^0)).$$

Hence $J(u) - J(u^0) > 0$ for any $u \in l^2(V), u \neq u^0$. It means that element u^0 is the unique minimum for the functional $J(u)$. The theorem is proved.

Further, it is directly verified that the mapping $\mathcal{V}^* : l^2 \to l^2$ defined by the formula

$$(\mathcal{V}^*\psi)(t-1) = \sum_{i=0}^{\infty} A_i^* \psi(t+i),\ t \in Z_+$$

is the conjugate operator for $\mathcal{V}$ where A_i^* are conjugate operators for A_i.

Consider the system of discrete equations

$$x(t+1) = \sum_{i=0}^{t} A_i x(t-i) - BR^{-1}B^* z(t+1),$$

$$z(t-1) = \sum_{i=0}^{\infty} A_i^* z(t+i) + Gx(t),\ t \in Z_+. \tag{27}$$

with the boundary conditions

$$x(0) = x_0,\ |z(t)| \to 0 \quad \text{as} \quad \text{t} \to \infty. \tag{28}$$

Theorem 9. *Let* $\sum_{i=0}^{\infty} |A_i| < 1$. *Then the boundary-value problem (27), (28) is solvable in* l^2.

Proof. According to the theorem 1 the equation (1) with $u(t) = u^0(t),\ t \in Z_+$, (where $u^0 \in l^2(V)$ is defined by (26)) yields the unique element $x^0 \in l^2(E)$. Setting $x(t) = x^0(t),\ t \in Z_+$ in the second equation in (27) we have that the function $z^0(t) = \sum_{i=0}^{\infty} Q_i^* G x^0(t+i),\ t \in Z_+$ is a solution of this equation. Here the set of operators $Q_t^* : E \to E$ is defined as follows $Q_0^* = I,\ Q_{t+1}^* = \sum_{i=0}^{t} A_i^* Q_{t-i}^*,\ t \in Z_+$. The fact that $z^0 \in l^2(E)$ can be proved by analogy with the theorem 1. Thus, the pair $(x^0(t), z^0(t))$ satisfies the second equation in (27) and the boundary conditions (28). In order to complete the proof of theorem it is sufficient to show that $u^0(t) = -R^{-1}B^* z^0(t+1), t \in Z_+$. Indeed, from (26) we have that $\mathcal{R}u^0 + L^*\mathcal{G}Lu^0 + L^*\mathcal{G}\omega = 0$. Since $x^0 = Lu^0 + \omega$ then $\mathcal{R}u^0 = -L^*\mathcal{G}x^0$. Here $L^* : l_0^2(E) \to l^2(E)$ is the operator conjugate with L defined by

$$(L^*y)(t-1) = B^* \sum_{i=0}^{\infty} Q_i^* y(t+i),\ t \in Z_+. \tag{29}$$

By definition of the function $z^0(t)$ we have

$$B^* z^0(t) = B^* \sum_{i=0}^{\infty} Q_i^* G x^0(t+i) = (L^* \mathcal{G} x^0)(t-1).$$

Hence $Ru^0(t) = -B^* z^0(t+1),\ t \in Z_+$ and the proof is completed.

Theorem 10. *The minimization problem for (1),(25) in the space* l^2 *has a unique optimal solution* $u^0(t)$, *which is given by*

$$u^0(t) = -R^{-1}B^* z(t),\ t \in Z_+,$$

where $z(t)$ *is the solution of (27), (28).*

Proof. The uniqueness of the optimal solution was proved in the theorem 8. Further, assume that the pair $(x(t), z(t)),\ t \in Z_+$ is a solution of the problem (27), (28). Consider the admissible control function given by $v(t) = -R^{-1}B^* z(t+1)$. Then from (27), (28) we have

$$x(t) = -\sum_{i=0}^{t-1} Q_i R^{-1} B^* z(t-i) + Q_t x_0,$$

$$z(t) = \sum_{i=0}^{\infty} Q_i^* G x(t+i),\ t \in Z_+. \tag{30}$$

In accordance with (29) the element $v \in l^2(V)$ is presented in the form $v = -R^{-1}L^*\mathcal{G}x$ or $\mathcal{R}v = -L^*\mathcal{G}x$. On the other hand similarly to the proof of the theorem 8 the first equation in (27) yields $x = \omega - LR^{-1}B^*z$ or $x = \omega + Lv$. Then the relations $\mathcal{R}v = -L^*\mathcal{G}x = -L^*\mathcal{G}x + L^*\mathcal{G}\omega - L^*\mathcal{G}(\omega = L^*\mathcal{G}(\omega - x) - L^*\mathcal{G}\omega = L^*\mathcal{G}Lv - L^*\mathcal{G}\omega$ hold. Hence, $v = -(\mathcal{R} - L^*\mathcal{G}L)^{-1}L^*\mathcal{G}\omega$. Thus, the element v is the same as element u^0 defined by (26). It means that $u^0(t) = -R^{-1}B^* z(t+1), t \in Z_+$ is the optimal solution. The theorem is proved.

6 Feedback control

As follows from Theorem 10 the optimal control for problem (1),(25) was represented by a linear function on the variables of the conjugate system. It is known that the optimal feedback control is defined via the algebraic Riccati equation for many optimal control problems. For considered problem the following result is true.

Theorem 11. *The discrete Fourier transformation* $V(\omega) = \sum_{t=0}^{\infty} u^0(t)e^{-i\omega t},\ \omega \in [0, 2\pi], i^2 = -1$ *of optimal control* $u^0(t)$ *for problem (1),(25) can be represented by* $V(\omega) = K(\omega)X(\omega)$. *Here* $X(\omega)$ *is the discrete Fourier transformation of the optimal trajectory* $x^0(t), t \in Z_+$,

$$K(\omega) = -[R + B'P(\omega)B]^{-1}B'P(\omega)\mathcal{U}(e^{-i\omega}),$$

$$\mathcal{U}(e^{-i\omega}) = \sum_{i=0}^{\infty} A_t e^{-i\omega t},\ \omega \in [0, 2\pi]$$

and $P(\omega)$, $\omega \in [0, 2\pi]$ is a solution of the following equation

$$e^{\omega}P(\omega) = G + \mathcal{U}^*(e^{-i\omega})P(\omega)[I - B[R+$$
$$+B'P(\omega)B]^{-1}B'P(\omega)]\mathcal{U}(e^{-i\omega}).$$

Moreover, the minimal value of cost functional (25) is equal to $J(u^0) = \frac{1}{2\pi}\int\limits_0^{2\pi} e^{\omega}(P(\omega)x_0, x_0)d\omega$. Proof. Applying the discrete Fourier transform to the equation (1) yields

$$(e^{-i\omega}I - \mathcal{U}(e^{-i\omega}))X(\omega) = BV(\omega) + e^{i\omega}x_0. \qquad (31)$$

Taking into account the known Parseval's equality the cost functional (25) on the solution of equation (1) can be written in the form

$$J(u) = \frac{1}{2\pi}\int\limits_0^{2\pi}[(X(\omega), GX(\omega)) + (V(\omega), RV(\omega)]d\omega.$$

Let $P(\omega)$, $\omega \in [0, 2\pi]$ be an arbitrary collection of self-adjoint nonnegative operators from E into E such that $\int\limits_0^{2\pi} |P(\omega)|d\omega < \infty$. It is immediately follows from (31) that the identity

$$0 = -e^{\omega}(P(\omega)X(\omega), X(\omega)) + e^{\omega}(P(\omega)x_0, x_0)+$$
$$(P(\omega)(\mathcal{U}(e^{-i\omega})X(\omega)+BV(\omega)), (\mathcal{U}(e^{-i\omega})X(\omega)+BV(\omega)))$$

holds for any $\omega \in [0, 2\pi]$.

Integrating the latter from 0 to 2π and adding with $J(u)$,we obtain then

$$J(u) = \frac{1}{2\pi}\int\limits_0^{2\pi}[e^{\omega}(P(\omega)x_0, x_0) - e^{\omega}(P(\omega)X(\omega), X(\omega))$$
$$+(GX(\omega), X(\omega)) + (RV(\omega), V(\omega))+$$
$$(P(\omega)\mathcal{U}(e^{-i\omega})X(\omega), \mathcal{U}(e^{-i\omega})X(\omega) + (P(\omega)BV(\omega)),$$
$$BV(\omega)) + (P(\omega)\mathcal{U}(e^{-i\omega})V(\omega), BV(\omega))]d\omega.$$

Now the functional $J(u)$ can be represented as

$$J(u) = \frac{1}{2\pi}\int\limits_0^{2\pi} e^{\omega}(P(\omega)x_0, x_0)d\omega + \frac{1}{2\pi}\int\limits_0^{2\pi}[(F(\omega)X(\omega),$$
$$X(\omega))d\omega + ((R + B'P(\omega)B)S(\omega), S(\omega))]d\omega,$$

where

$$S(\omega) = V(\omega) + [R + B'P(\omega)B]^{-1}B'P(\omega)\mathcal{U}(e^{-i\omega})X(\omega),$$
$$F(\omega) = G - e^{\omega}P(\omega) - \mathcal{U}^*(e^{-i\omega})P(\omega)\mathcal{U}(e^{-i\omega})-$$
$$-\mathcal{U}^*(e^{-i\omega})P(\omega)[R + B'P(\omega)B]^{-1}B'P(\omega)\mathcal{U}(e^{-i\omega}),$$

Now we suppose that the set of the operators $P(\omega)$, $\omega \in [0, 2\pi]$ satisfies to the conditions $F(\omega) = 0$, $\omega \in [0, 2\pi]$ and that leads to the equation being required. On noting this we receive

$$J(u) = \frac{1}{2\pi}\int\limits_0^{2\pi} e^{\omega}(P(\omega)x_0, x_0)d\omega+$$
$$+\frac{1}{2\pi}\int\limits_0^{2\pi}((R + B'P(\omega)B)S(\omega), S(\omega))d\omega.$$

Since $R > 0$, $P(\omega) \geq 0, \omega \in [0, 2\pi]$ then the minimal value of $J(u)$ is equal to $J^0 = \frac{1}{2\pi}\int\limits_0^{2\pi} e^{\omega}(P(\omega)x_0, x_0)d\omega$ that is attained in the case $S(\omega) = 0$, i.e. for $V(\omega) = K(\omega)X(\omega)$. The theorem is proved.

7 Conclusion

As known Riccati equations play a key role for many problems. We consider the solvability problem for Riccati type equation given by theorem 11

$$\mathcal{F}(P) \equiv e^{-w}(G + \mathcal{U}^*(e^{-i\omega})P(\omega)[I - B[R+$$
$$+B'P(\omega)B]^{-1}B'P(\omega)]\mathcal{U}(e^{-i\omega})) = P. \qquad (32)$$

Theorem 12. *For any $w \in [0, 2\pi]$ equation (32) has a unique nonnegative self-adjoint stabilizing solution $P(w)$ if and only if the pair $(\mathcal{U}(e^{-iw}), B)$ is stabilize and the kernel of matrix G has not any eigenvector corresponding to eigenvalues λ with $|\lambda| = 1$. If these conditions are held then the solution of equation (32) can be determined by the successive approximations methods $P_{k+1} = \mathcal{F}(P_k)$ starting from any initial point P_0 satisfying the condition $ker P_0 \subset L$, where L is the maximal subspace that belongs to the null-space of G and that is invariant to $\mathcal{U}(e^{-iw})$.*

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References

[1] M. Dymkov and I. Gaishun, "Linear-quadratic Optimal Control Problem for a Discrete Volterra Equation", Bul. Pol. Acad. Sc. Tech. Sc., 1997, v. 45, No 2, pp. 251-262.

[2] S. Eladi, "Periodicy and Stability of Linear Volterra Systems", J. Math. Anal. Appl., Vol. 181, No 2, 1994, pp.483-492.

[3] I. V. Gaishun, " Stability for discrete separable Volterra equations", Diff.equat., 1996, v. 32, No 6, pp. 739-746.

[4] V.B. Kolmanovskii, L.E. Shaikhet, "Asymptotic stability for some discrete systems with inclusions", Avtom. and Telemech., 1996, No 12, pp. 58-66 (in Russian).